Infinite Products & Zero-One Laws

In categorical probability

(Joint work w. Tobias Fritz)

Summary

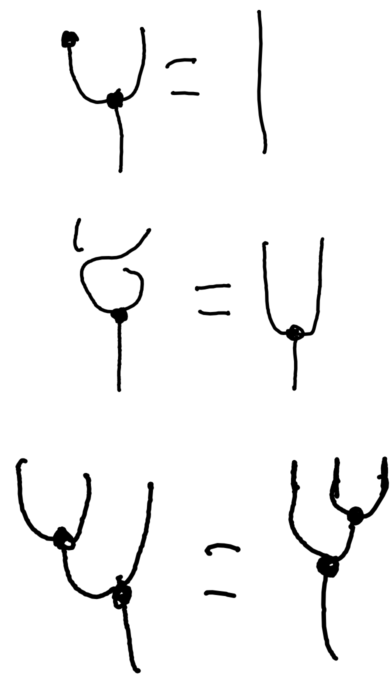
- The framework: "Markov categories"
- Probabilistic notions
- What are infinite tensor products?
- Kolmogorov's 0-1 law, categorically.

The Framework: Markov Categories

A <u>Markov category</u> is:

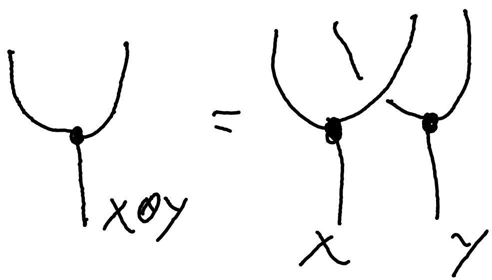
- A symmetric monoidal category $(\mathcal{C}, \otimes, I)$
- With a commutative comonoid on each object

$\mathrm{copy}_X : X \to X \otimes X$ $\qquad$ $\mathrm{del}_X : X \to I$



- I is terminal
- "compatible w sym. mon. str"

eg:



Markov cats

Examples

Stoch = measurable spaces
"markov kernels", ie maps $X \to \left\{ \begin{matrix} \text{distributions} \\ \text{on } Y \end{matrix} \right\}$
$X \otimes Y := X \times Y$ (but not cartesian!)

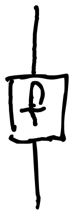
FinStoch: Stoch but only finite sets

BorelStoch: Stoch but only "standard Borel spaces"
("nice" measurable spaces)

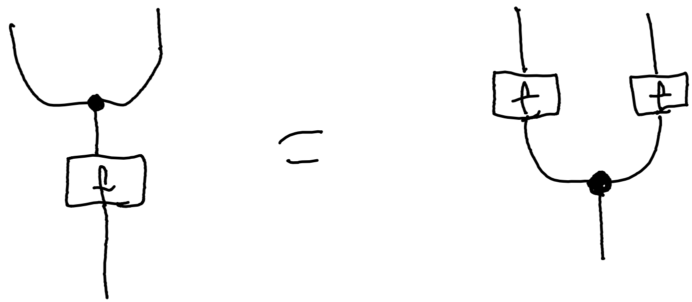
$(\mathcal{C}, \times, *)$ if $\mathcal{C}$ has products.

The general principle:
Spaces and maps <u>with randomness</u>

Probabilistic notions



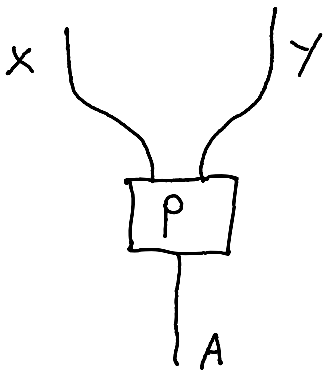
is <u>deterministic</u>

if 

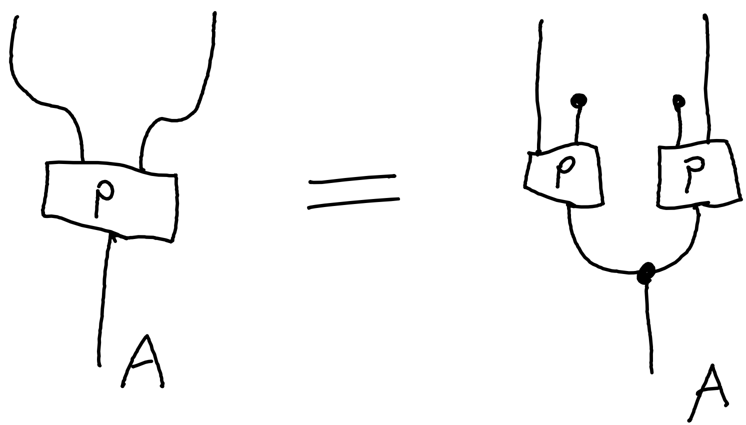
Have subcat $\mathcal{C}_{det} \subseteq \mathcal{C}$ on deterministic maps

$\mathcal{C}$ is Cartesian $\iff$ all f deterministic

i.e "no randomness".



makes X, Y independent given A, $X \perp Y \mid A$, if:



And many other notions!

- conditional distribution
- conditional independence
- sufficient statistics
- informativeness preorder on experiments (arxiv:2010.07416)

$\vdots$

Kolmogorov's 0-1 law:

Given random variables V_i which are all independent, and an event E which depends "measurably" on V_i (formally: $E \in \sigma(V_1, V_2, \ldots)$), and s.t $E \perp (V_1, \ldots V_N)$ for all $N \in \mathbb{N}$, then $P(E) \in \{0, 1\}$.

How to accomodate this?
Need a notion of <u>infinite</u> <u>product</u>.

Infinite products

Let X_j, $j \in J$ be a family of objects of $\mathcal{C}$.

If J finite, no problem defining $\bigotimes_{j \in J} X_j$ (up to canonical iso etc).

What about infinite J?

Idea: Take the limit

$$\varprojlim_{F \subset J} \bigotimes_{j \in F} X_j$$

F finite

with

$$\bigotimes_{j \in F} X_j \longrightarrow \bigotimes_{j \in F'} X_j$$

for $F' \subseteq F$ induced by deletions $X_j \to I$

∴ This may be poorly behaved.

Need to impose:

- $\left(\lim_F \bigotimes_{j \in F} X_j\right) \otimes Y = \lim_F \left(\bigotimes_{j \in F} X_j \otimes Y\right)$

  (i.e limit is preserved by $- \otimes Y$ for all Y)

- The maps $\lim_F \bigotimes_{j \in F} X_j \longrightarrow \bigotimes_{j \in F} X_j$

  are deterministic.

If so we call this limit a <u>Kolmogorov product</u> and write $\bigotimes_{j \in J} X_j$

Examples

If $(\mathcal{C}, \times)$ cartesian monoidal and $\prod_{i \in J} X_i$ exists, it is a Kolmogorov product.

Stoch does not have all Kolm. prods, not even countable ones, since the Kolmogorov extension thm is false in general.

BorelStoch has all countable Kolmogorov products.

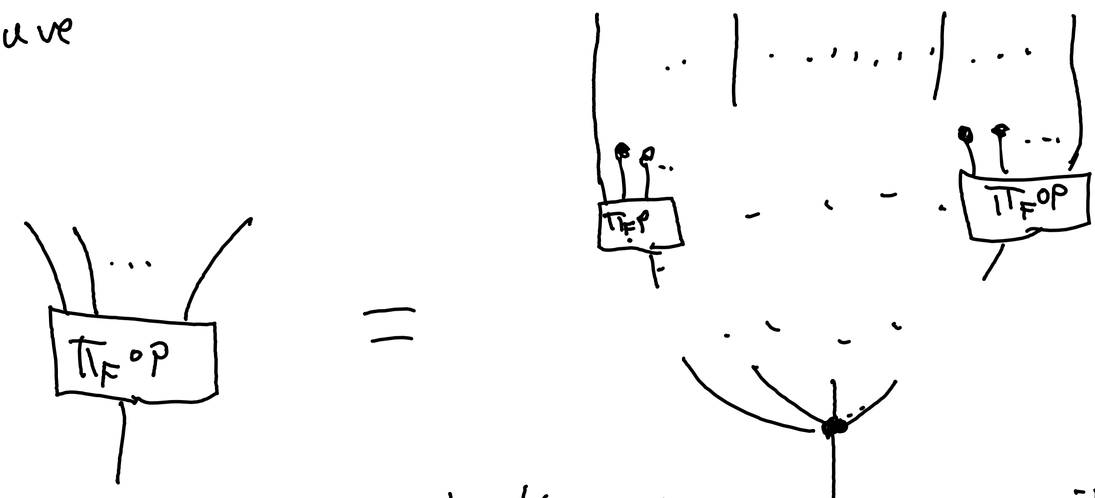
One last ingredient:

Let $p: A \longrightarrow \bigotimes_{j \in J} X_j$. Say all the X_j are indep.

given A, or $\underset{j \in J}{\perp} X_j \mid A$

if for each projection $\pi_F : \bigotimes_{j \in J} X_j \longrightarrow \bigotimes_{j \in F} X_j$,

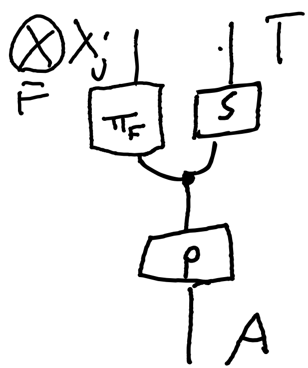
have



(ie for all finite subsets, those coords are indep.)

Thm (Fritz - R)

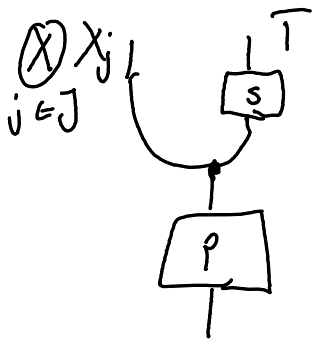
Let J be an infinite set. Let X_j, $j \in J$ be a family of objects s.t $\bigotimes_{j \in J} X_j$ exists. Let $p : A \longrightarrow \bigotimes_{j \in J} X_j$ be a map s.t $\perp_{j \in J} X_j \,|\, A$. Let $s : \bigotimes_{j \in J} X_j \longrightarrow T$ be deterministic. Assume for all $F \subseteq J$ finite, the map



makes $\bigotimes_F X_j \perp T \,|\, A$. Then $sp : A \longrightarrow T$ is deterministic.

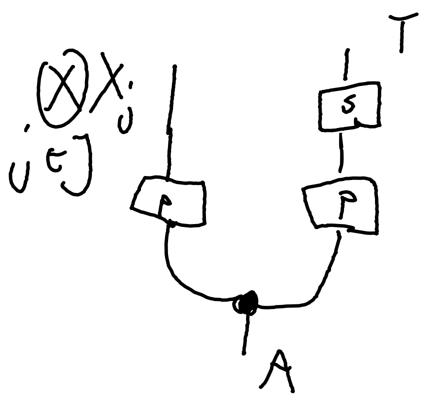
Proof

First I claim that in



we have $\bigotimes_{j\in J} X_j \perp T \mid A$.

To see this we must show it equals

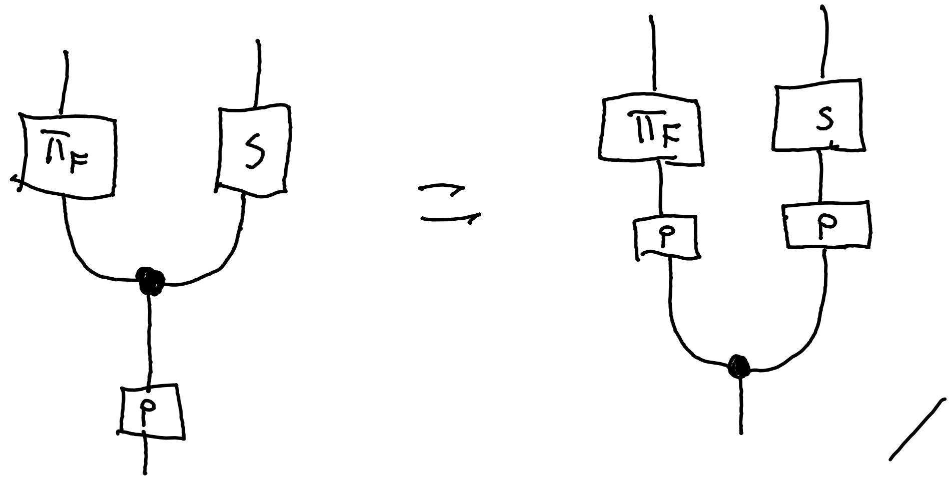


We compare two maps into $\lim_{F\subseteq J} \bigotimes_F X_j \otimes T$

$$\cong \lim_{F\subseteq J}\left(\bigotimes_F X_j \otimes T\right).$$

So ~~lets the components $A \to \bigotimes X_j \otimes T$ agree~~

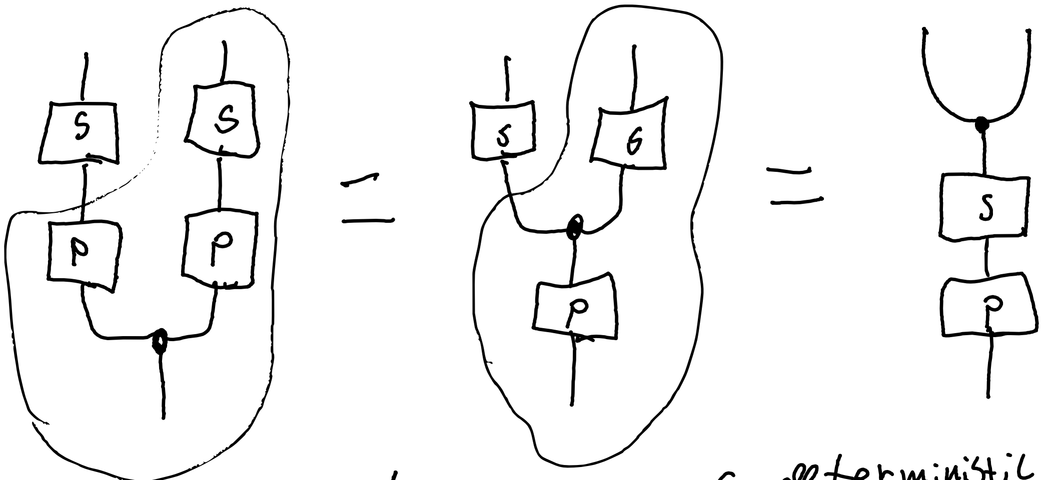
In other words, must show



i.e $\bigotimes_F X_j \perp T \mid A$.

This holds by assumption!

Now calculate:



apply independence to this part

s deterministic

This means sop is deterministic!

To recover Kolmogorov's Thm:

$\mathcal{C}$ = Borel Stoch, $J = \mathbb{N}$

$X_i = \mathbb{R}$ $\forall i$

$A = \{*\}$, p = joint distribution of vars

$T = \{0,1\}$, s = indicator of event

$sp(*)$ is dist on $\{0,1\}$. Probability of 1

= probability of event.

If **deterministic**, this is 0 or 1

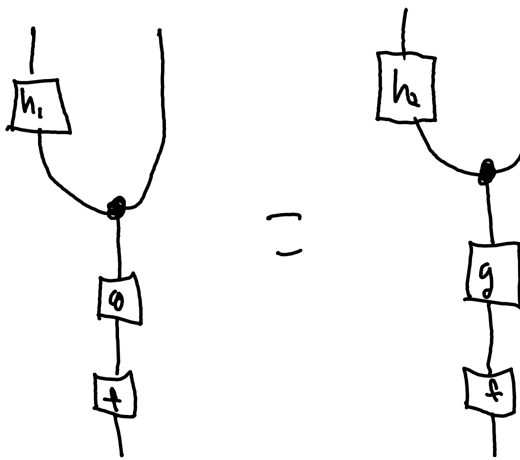
Hewitt-Savage 01 law

Given a family of random variables V_i, J infinite, independent and identically distributed, and given an event E which is in $\sigma(V_1, \ldots)$, and so that E independent of finite permutations of the V_j, $P(E) \in \{0,1\}$.

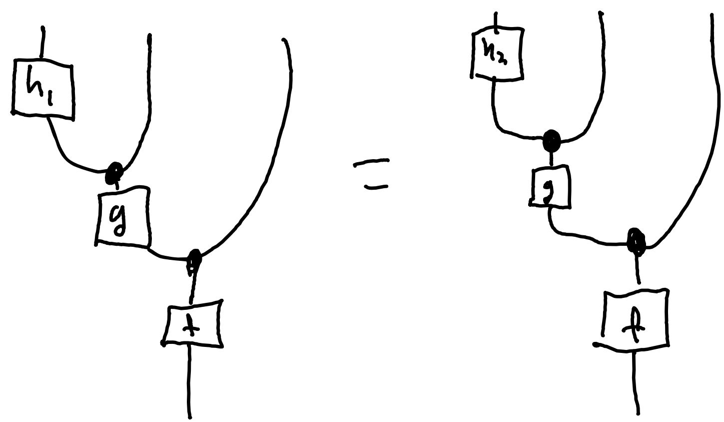
Example: $E =$ "$V_i = 0$ for some i"

To prove, need this axiom

A Markov category $\mathcal{C}$ is *causal* if, whenever



then also



Abstract HS (Fritz - R)

Let X be an object, and let $X^J = \bigotimes_{j \in J} X$ be a "Kolmogorov power".

Any function $\sigma : J \to J$ acts on this in a natural way - informally, we can describe as $(X_j)_{j \in J} \mapsto (X_{f(j)})_{j \in J}$. Denote that map $\hat{\sigma} : X^J \to X^J$.

Let $p : A \to X^J$ and $s : X^J \to T$ be maps s.t

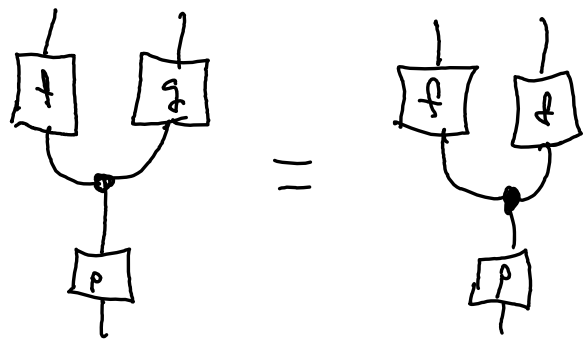
- The X's are independent given A
- s is deterministic
- for any finite permutation σ,

$$\hat{\sigma} p = p, \quad s \hat{\sigma} = s$$

And suppose $\mathcal{C}$ causal. Then sp is deterministic

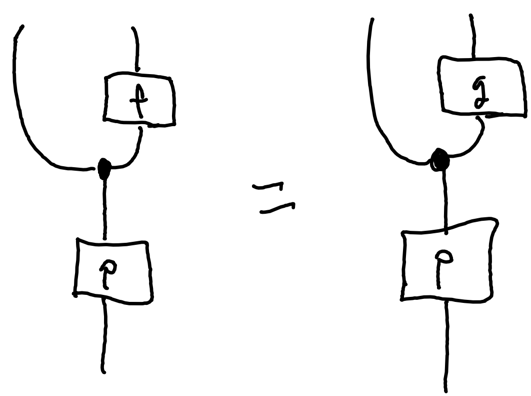
Lemma

Let $\mathcal{C}$ be causal and suppose

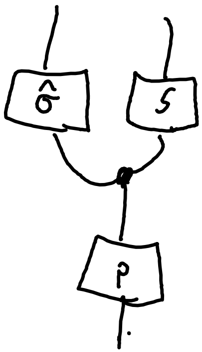


w/ f deterministic.

then



Pf of Thm



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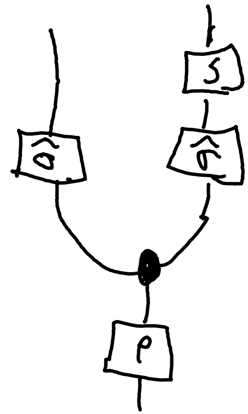

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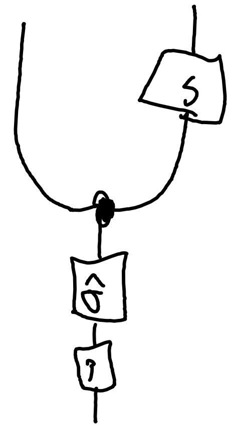

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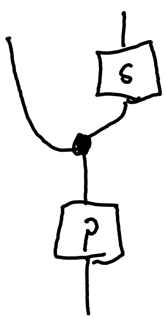
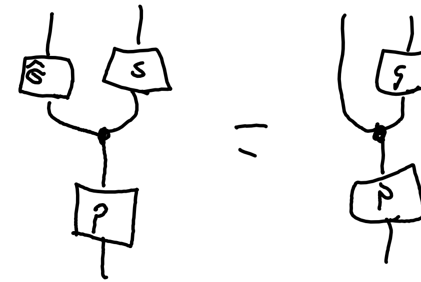


for $\sigma : J \to J$ finite permutation.

Pf, continued

I claim we have



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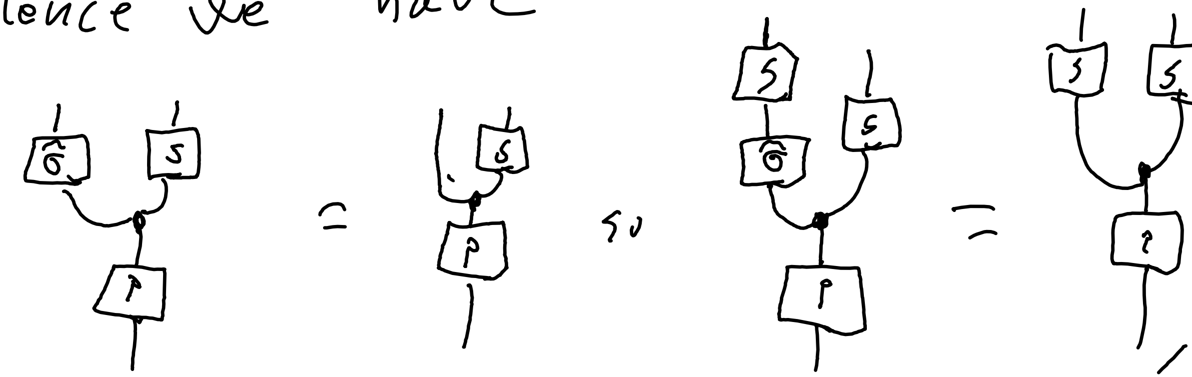
even for $\sigma : J \hookrightarrow J$ an injection.

Since $X^J \otimes T = \lim_F \left(\bigotimes_F X \otimes T \right)$, enough to show for finite marginals.

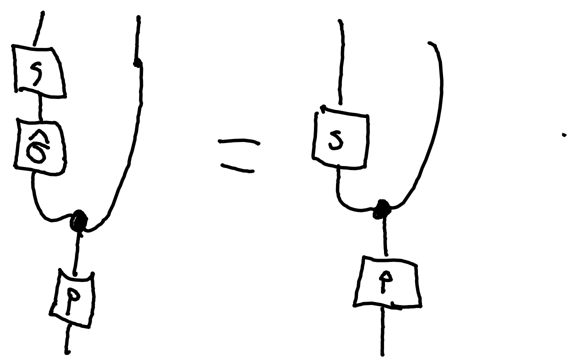
But for any finite F, i can find $\sigma' : J \to J$ a finite permutation s.t. $\sigma'|_F = \sigma|_F$. So this follows from the case where σ is a finite permutation

Pf, cont

Hence we have



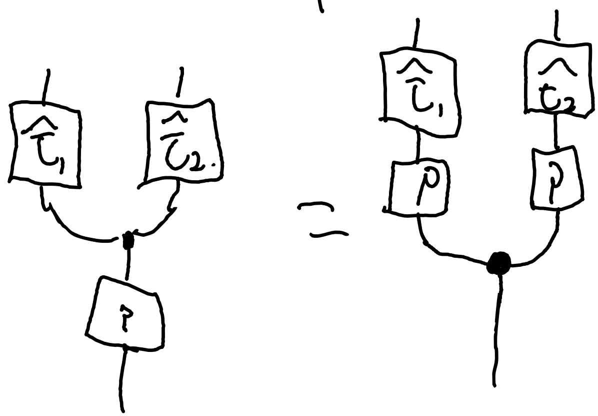
So lemma gives



Pf, cont

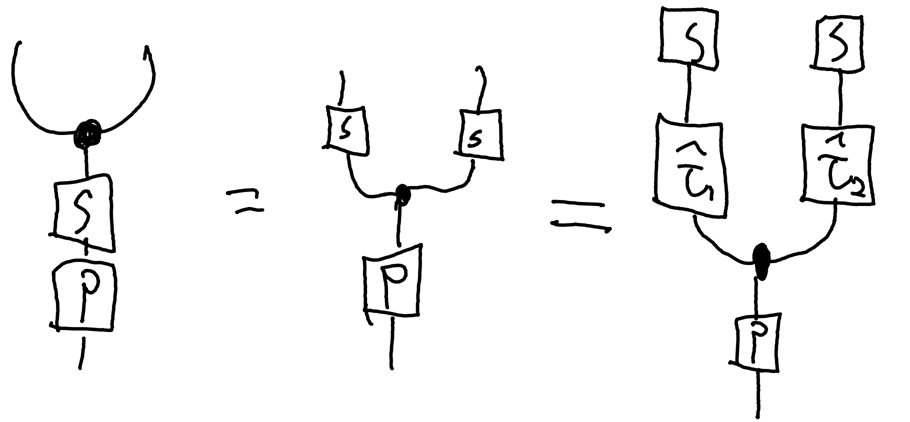
Let $J = J_1 \sqcup J_2$ be decomposition of J into subsets of equal cardinality to J. (here we use J infinite).

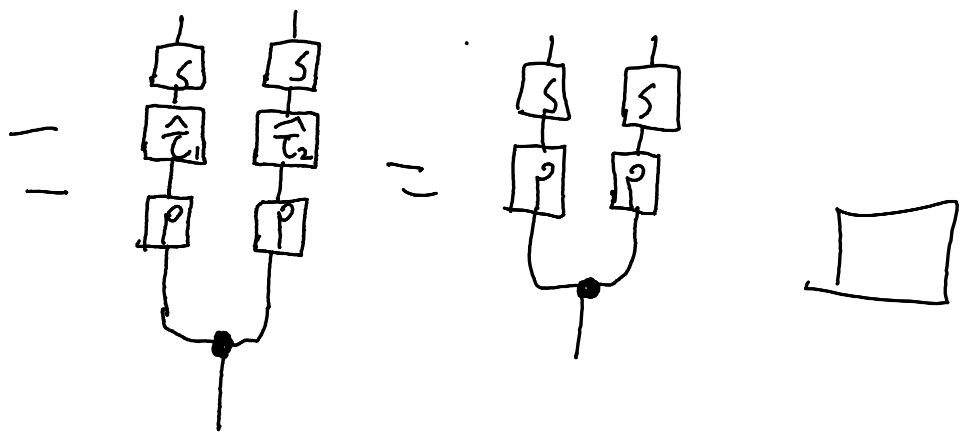
Let $\tau_1, \tau_2 : J \to J$ be injections w. image J_1, J_2. Then we have



(follows from indipendence)

putting things together:





THANK YOU
FOR
LISTENING!